COURSE NOTES

Logic and Arithmetic: Incompleteness

Part III of

Advanced Course in Foundations of Mathematics

A self-contained minicourse of approximately eight hours

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Introduction

These notes originated as the third part of Advanced Course in Foundations of Mathematics. They explore formal arithmetic, mathematical proof, and the limits of axiomatic theories. Although developed within a larger course, **the notes are self-contained**: they can also serve as the basis for an independent **eight-hour minicourse**. No prior study of Parts I and II is required.

Aims and organisation

The guiding questions are what can be expressed about the natural numbers, and what can be proved from a precise collection of axioms. The material is organised into three chapters.

1. **Hilbert's programme and arithmetic.** Historical motivation, the language and axioms of Peano arithmetic, and the distinction between formal derivability and truth in the standard model.
2. **The First Incompleteness Theorem.** Effective axioms, Gödel numbering, the fixed-point lemma, arithmetised provability, and Rosser's improvement of Gödel's argument.
3. **Consistency and independence.** The Second Incompleteness Theorem, Gentzen's consistency proof, Goodstein sequences, and the Kirby–Paris theorem.

A possible eight-hour schedule devotes about two hours to Chapter 1, three and a half to Chapter 2, and two and a half to Chapter 3. The detailed coding examples and some historical or proof-theoretic remarks are optional in a more concise presentation.

Prerequisites and use

Familiarity with elementary mathematics, sets, functions, and mathematical proofs is sufficient. Some acquaintance with logical connectives and quantifiers is helpful, but the necessary formal language is introduced in Chapter 1. The basic ideas of deduction and logical completeness are recalled when needed. Selected advanced arguments are labelled proof ideas, to distinguish them from complete derivations. Exercises at the end of each chapter support independent study, while the bibliography offers sources for further exploration.

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1

Hilbert's programme and arithmetic

The end of the nineteenth century and the beginning of the twentieth century saw an intense debate about the foundations of mathematics. New methods and abstract concepts had transformed the discipline, while paradoxes raised questions about the legitimacy of some forms of reasoning. New proof methods emerged, including:

— indirect existence proofs (e.g. Hilbert's basis theorem),[1]

— explicit occurrences of abstract objects (such as the new abstract notion of function),

— Cantor's theory of **transfinite numbers**,

— Zermelo's proof that every set can be well-ordered, using the axiom of choice.

This situation caused uncertainty regarding the foundations of mathematics: Brouwer rejects the Law of Excluded Middle, and Weyl criticizes impredicative set formation principles.

As a solution, in 1920 Hilbert proposed to formalise mathematical theories by means of precisely specified axioms and rules, and establish their consistency using finitistic methods. Hilbert proposed that the consistency of more complicated systems, such as real analysis, could be proven in terms of simpler systems. Ultimately, **the consistency of all of mathematics could be reduced to basic arithmetic**.

The main goal of Hilbert's programme was to provide secure foundations for all mathematics. For short its goals were:

— Codify mathematical reasoning in formal theories with effectively specified axioms and rules of inference.

— Prove the consistency of the formalised theories by finitistic means, i.e. without referring to the actual infinite and excluding every "objectionable" proof method.

More precisely, the programme had the goal of proving the following:

— **axiomatisability**: all mathematical statements can be written in a precise formal language, and manipulated according to well defined rules,

— **completeness**: all true mathematical statements can be proved in the formalism,

— **consistency**: no contradiction can be obtained in the formalism of mathematics,

[1] Actually, the first known proof of this kind is probably Gauss' 1799 proof of the Fundamental Theorem of Algebra.

— **conservation**: any result about "real objects" (such as natural numbers, points of the Euclidean plane, finite strings of symbols, etc.) obtained using reasoning about "ideal objects" (such as uncountable sets, transfinite numbers, points at infinity, etc.) can be proved without using ideal objects,

— **decidability**: there is an algorithm for deciding the truth or falsity of any mathematical statement.

Because the consistency of mathematics was at stake, Hilbert had required that the allowed methods be "absolutely reliable". He then identified such methods as those that can be seen as "symbolic manipulations of concrete signs according to specific mechanical rules". The basic acts in such manipulation consist in identifying two sequences of symbols as equal, in identifying a sequence of symbols as having a specific form, and in identifying a number of given sequences of symbols as an instance of a rule for the construction of a proof.

A first positive result for 'pure logic' is given by Gödel:

Theorem 1.1 (Soundness and Completeness)**.** For any formula A and set of formulae Γ in a given language $\mathscr{L}$,[a]

$$\Gamma \vdash A \iff \Gamma \vDash A.$$

[a] The $\Leftarrow$ direction is **completeness**, the $\Rightarrow$ direction is **soundness**.

The completeness theorem stated above concerns **first-order logic**: semantic consequence from a set of premises coincides with formal derivability from those premises. It does **not** assert that **PA** proves every sentence true in its standard model; this difference is central to the chapters that follow.

1.1 Peano Arithmetic

In **elementary number theory** or **arithmetic** one studies the properties of natural numbers with respect to addition and multiplication. We present a formal language $\mathscr{L}$ (**Language for Arithmetic**), rich enough to express properties of addition and multiplication of natural numbers.

Definition 1.2 (Formal language for Arithmetic)**.** The formal language $\mathscr{L}$ for arithmetic contains the following symbols:

— Logical symbols:

- Variables $a_0, a_1, a_2, \ldots$.
- The connectives $\rightarrow$ ('if ..., then ...'), $\wedge$ ('and'), $\vee$ ('or').
- The quantifiers $\forall n$ ('for all n, ...') and $\exists n$ ('there is at least one n such that ...').

— Non-logical symbols:

- The equality symbol $\equiv$ (interpreted as genuine identity, with the usual logical rules for equality).
- The constant c_0 ('zero').
- The unary function symbol s ('successor').[a]
- The binary function symbols $\oplus$ ('plus') and $\otimes$ ('times').

— Typographical symbols:

  – Parentheses (,).

[a] Instead of introducing an individual constant c_n for each individual natural number n, we can take only one individual constant c_0 together with a unary function symbol s, to be interpreted as the successor function. Then $s(c_0)$ can play the role of c_1, $s(s(c_0))$ can play the role of c_2, and so on.

Remark 1.3. Instead of $\equiv(r,s)$ one usually writes $r \equiv s$. We do not take $\bot$ ('falsity') as a primitive symbol, as we can define

$$\bot := c_0 \equiv s(c_0).$$

Then, as usual, we define $\neg$ ('not') and $\not\equiv$ ('unequal') as

$$\neg A := A \to \bot,$$
$$r \not\equiv s := \neg(r \equiv s).$$

Also $\leftrightarrow$ ('if and only if') is defined as an abbreviation:

$$A \leftrightarrow B := (A \to B) \wedge (B \to A).$$

Definition 1.4 (Terms of $\mathscr{L}$)**.** The **terms** of the language $\mathscr{L}$ for formal arithmetic are defined as follows:

— Each variable $a_0, a_1, a_2, \ldots$ is a term.

— c_0 is a term.

— If r and t are terms, then $s(r)$, $(r \oplus t)$ and $(r \otimes t)$ are also terms.

If no confusion is possible, parentheses are omitted as much as possible.

Example. Examples of terms of $\mathscr{L}$:

1. c_0,
2. a_1,
3. $c_0 \oplus a_1$,
4. $(c_0 \oplus a_1) \otimes a_1$,
5. $a_1 \otimes a_2$,
6. $c_0 \oplus (a_1 \otimes a_1)$,
7. $(c_0 \otimes a_1) \oplus (a_2 \otimes a_3)$.

Definition 1.5 (Formulae of $\mathscr{L}$)**.** If r and s are terms, then $\equiv(r,s)$ is an **atomic formula** of the language $\mathscr{L}$ for formal number theory. More generally, **formulae** of $\mathscr{L}$ are inductively defined as follows:

— Every atomic formula of $\mathscr{L}$ is a formula of $\mathscr{L}$.

— If A and B are formulae of $\mathscr{L}$, then also $(A \to B), (A \wedge B), (A \vee B)$ are formulae of $\mathscr{L}$.

— If A is a formula of $\mathscr{L}$ and x is a variable, then also $\forall x(A)$ and $\exists x(A)$ are formulae of $\mathscr{L}$.

Example.

1. $\forall x \forall y\,((x+y)\cdot(x+y) = (x\cdot x) + (0'' \cdot x \cdot y) + (y \cdot y))$.

   This is read as 'for all natural numbers n and m, (n plus m) times (n plus m) equals (n times n) plus two times (n times m) plus (m times m)'.

2. $\forall x \forall y\,((x = y) \to (x \cdot x = y \cdot y))$.

   This is read as 'for all natural numbers n and m, if n equals m, then n square equals m square'.

3. $\forall x(x + 0 = x)$.

   This is read as 'for all natural numbers n, n plus zero equals n'.

4. $\exists x(x \cdot x = x)$.

   This is read as 'there is at least one natural number n such that n square equals n'.

So far we have introduced a (first-order) formal language $\mathscr{L}$ for elementary number theory, in which propositions about addition and multiplication of natural numbers can be formulated. The next step is to select a number of arithmetic (non-logical) axioms, formulated in this language, in order to be able to deduce formally properties of natural numbers. To that purpose Giuseppe Peano formulated in 1889 the following set **PA** of arithmetic axioms, named after him. The Peano axioms are formulae in the formal language $\mathscr{L}$ for elementary number theory, and they are true in the intended interpretation. The induction axiom schema yields an induction axiom for any formula A in the language $\mathscr{L}$.

Definition 1.6 (Peano axioms)**.** We denote by **PA** the set of the **Peano axioms**:[a]

— an (in fact logically valid) equality principle $\equiv$:

$$\forall x \forall y \forall z(x \equiv y \wedge x \equiv z \to y \equiv z), \tag{Ax. 1}$$

— axioms for the successor function $'$:

$$\forall x \forall y(x \equiv y \leftrightarrow s(x) \equiv s(y)), \tag{Ax. 2}$$
$$\forall x(c_0 \not\equiv s(x)), \tag{Ax. 3}$$

— axioms for addition $\oplus$:

$$\forall x(x \oplus c_0 \equiv x), \tag{Ax. 4}$$
$$\forall x \forall y(x \oplus s(y) \equiv s(x \oplus y)), \tag{Ax. 5}$$

— axioms for multiplication $\otimes$:

$$\forall x(x \otimes c_0 \equiv c_0), \tag{Ax. 6}$$
$$\forall x \forall y(x \otimes s(y) \equiv (x \otimes y) \oplus x). \tag{Ax. 7}$$

— axiom schema of **Peano induction**:

$$(E(c_0) \wedge \forall x(E(x) \to E(s(x)))) \to \forall x\, E(x) \tag{Ax. 8}$$

for every predicate E.[b]

[a] This is a refined version of the list of axioms of arithmetic first introduced by Peano in [20]. Notably, he did not consider 0 to be a natural number, and started from 1 instead.

[b] This is an axiom schema. Peano's original formulation was a second-order statement.

Definition 1.7 (Standard model of arithmetic)**.** The model $\mathscr{N} = \langle \mathbb{N}, =, 0, ', +, \cdot \rangle$ which interprets

— the individual variables as natural numbers (i.e., as elements of $\mathbb{N}$),

— the symbol $\equiv$ as the equality relation $=$ between natural numbers,

— the symbol c_0 as the natural number 0,

— the symbol s as the successor function $' \colon \mathbb{N} \to \mathbb{N}$ defined by $n' := n + 1$,

— the symbol $\oplus$ as addition $+$ of natural numbers,

— the symbol $\otimes$ as multiplication $\cdot$ of natural numbers,

is the **standard model of arithmetic**.

One usually writes $r = s$ instead of $=(r, s)$, and $r \neq s$ instead of $\neg(r = s)$. Individual natural numbers are introduced as usual:

$$1 := 0', \qquad 2 := 1', \qquad 3 := 2', \qquad 4 := 3', \qquad 5 := 4', \qquad 6 := 5' \qquad \ldots$$

Remark 1.8. Consider the sentence $\forall x(x \oplus c_0 \equiv x)$. It is true in the standard model $\mathscr{N}$, because there $\oplus$ denotes addition and c_0 denotes zero. It is not logically valid, however. For example, let $\mathscr{M}$ have domain $\mathbb{Q}$; interpret $\equiv$ as ordinary equality, c_0 as 5, s as the map $q \mapsto q + 1$, $\oplus$ as subtraction, and $\otimes$ as multiplication. The sentence then asserts $\forall q \in \mathbb{Q}(q - 5 = q)$, which is false.

The symbols c_0, s, $\oplus$ and $\otimes$ belong to the **object language**, whereas 0, successor, $+$ and $\cdot$ denote mathematical objects and operations in the intended interpretation. Their meanings can vary from one structure to another. In contrast, logical equality $\equiv$ is always interpreted as identity. When the interpretation is clear, we use the familiar symbols 0, $'$, $+$, $\cdot$ and $=$ also within the formal language.

Remark 1.9. By Gödel's Completeness Theorem we know that for any formula A in $\mathscr{L}$,

$$\mathbf{PA} \vdash A \iff \mathbf{PA} \vDash A.$$

In particular, since $\mathscr{N}$ is a model of **PA**:

$$\mathbf{PA} \vdash A \implies \mathscr{N} \vDash A.$$

It naturally arises the question of whether the converse implication holds, i.e.

$$\mathbf{PA} \vdash A \impliedby \mathscr{N} \vDash A?$$

It will be clear from Gödel's first incompleteness theorem that this is not the case.

Exercises

Exercise. Prove that

1. $\mathbf{PA} \vdash \forall x(x = x)$,
2. $\mathbf{PA} \vdash \forall x \forall y(x = y \to y = x)$,
3. $\mathbf{PA} \vdash \forall y \forall x(x' + y = (x + y)')$,
4. $\mathbf{PA} \vdash \forall x \forall y(x + y = y + x)$.

Proof (Solution of (i).).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x(x+0=x)}\,\text{Ax. 4}}{n+0=n}\,\forall^-}{n+0=n \wedge n+0=n}\,\wedge^+ \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall x \forall y \forall z(x=y \wedge x=z \to y=z)}\,\text{Ax. 1}}{\forall y \forall z(n+0=y \wedge n+0=z \to y=z)}\,\forall^-}{\forall z(n+0=n \wedge n+0=z \to n=z)}\,\forall^-}{n+0=n \wedge n+0=n \to n=n}\,\to^-$$

$$\dfrac{n=n}{\forall x(x=x)}\,\forall^+$$

■

Exercise. Find formulae A_1, A_2, A_3 and interpretations $\mathscr{M}_1, \mathscr{M}_2$ such that

1. $\mathscr{N} \vDash A_1$ and $\mathscr{M}_1 \nvDash A_1$,
2. $\mathscr{N} \nvDash A_2$ and $\mathscr{M}_2 \vDash A_2$,
3. $\vDash A_3$.

2

Gödel's First Incompleteness Theorem

Definition 2.1 (Consistent, Axiomatisable)**.** Let Γ be a set of formulae (in $\mathscr{L}$ or in any other language). Γ is said to be **consistent** if there is no formula A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.

Let Γ be a set of formulae (in $\mathscr{L}$ or in any other language). Γ is said to be **axiomatisable** if there is a subset $\Gamma_0 \subseteq \Gamma$ such that:

1. Γ_0 is **decidable**, i.e., there is a decision method which decides for any formula A in the language whether A is in Γ_0 or A is not in Γ_0,
2. for any formula A in the language, $\Gamma_0 \vdash A$ if and only if $\Gamma \vdash A$.

The elements of Γ_0 are called **axioms** for Γ.

Theorem 2.2 (Gödel–Rosser First Incompleteness Theorem for Arithmetic)**.** Let $\mathbf{T}$ be a consistent and axiomatisable extension of **PA**. Then there is a closed formula $A_{\mathbf{T}}$ in $\mathscr{L}$ such that

1. $A_{\mathbf{T}}$ is **true** in the intended interpretation, i.e.,

$$\boxed{\mathscr{N} \vDash A_{\mathbf{T}},}$$

2. $A_{\mathbf{T}}$ is **undecidable** on the basis of $\mathbf{T}$, i.e.,

$$\boxed{\mathbf{T} \nvdash A_{\mathbf{T}} \quad \text{and} \quad \mathbf{T} \nvdash \neg A_{\mathbf{T}}.}$$

The theorem says that an effectively axiomatised consistent extension $\mathbf{T}$ of **PA** cannot decide every arithmetical sentence. The stated consistency-only version includes Rosser's 1936 improvement [21] of Gödel's original proof. The distinction matters: the straightforward Gödel sentence yields non-provability from consistency, but ruling out a proof of its negation by the same argument requires an additional assumption such as 1-consistency. We make both steps explicit below. Before giving the proof, we observe a few remarks.

Remark 2.3. Notice that the three conditions on the theory $\mathbf{T}$ are necessary. Let us

consider the following theories that satisfy only two of the three conditions:

T	consistent	axiomatisable	extension of PA
collection of all the true formulae in $\mathscr{N}$	✓	✗	✓
collection of all the formulae in $\mathscr{L}$	✗	✓	✓
pure logic	✓	✓	✗

These examples show why each hypothesis in the theorem matters: the asserted conclusion need not hold when a hypothesis is dropped.

Remark 2.4. Since the set **PA** of Peano's axioms satisfies the conditions in Theorem 2.2, Gödel's incompleteness theorem says in particular that there is a formula A_1 which is true in the intended interpretation, but which cannot be deduced from **PA**, in symbols, $\mathbf{PA} \nvdash A_1$.

Because A_1 is true in the intended interpretation, we might extend **PA** with the formula A_1 to the set $\mathbf{PA} \cup \{A_1\}$. But then, taking $\mathbf{T} := \mathbf{PA} \cup \{A_1\}$, Gödel's incompleteness theorem says that there is a formula A_2, depending on $\mathbf{PA} \cup \{A_1\}$, such that A_2 is true in the intended interpretation and $\mathbf{PA}, A_1 \nvdash A_2$.

In a similar way, we can find a formula A_3 such that A_3 is true in the intended interpretation and such that $\mathbf{PA}, A_1, A_2 \nvdash A_3$, and so on. Thus **PA** is not only incomplete but, in a quite crucial sense, is **incompletable**.

Remark 2.5. Consider a set theory (e.g. **CZF**, **ZF**, **ZFC**, etc.). Starting with the empty set $\emptyset$, form the sequence

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$$

This sequence has the structure of the natural numbers. We can pick out a first member (corresponding to zero); each member has one and only one successor; it never repeats. We can go on to define analogues of addition and multiplication.

Now, any standard set theory allows us to define this sequence. So if we could have a complete axiomatisable set theory, then we could, in particular, have a negation-complete theory of the fragment of set theory which provides us with an analogue of arithmetic. Adding a simple routine for translating the results for this fragment into the familiar language of **PA** would then give us a complete sound theory of **PA**. But Gödel's First Incompleteness Theorem (2.2) tells us there can't be such a theory. So there cannot be a complete axiomatisable set theory.

More generally, the result applies to consistent effectively axiomatised theories that can interpret enough arithmetic. The precise hypothesis is that the theory can represent the required elementary computations; the mere presence of infinitely many objects is not sufficient.

Remark 2.6 (Isaacson's Thesis)**.** If we are to give a proof of any **true** sentence of $\mathscr{N}$ which is independent of **PA**, then we will need to appeal to ideas that go beyond those which are constitutive of our understanding of basic arithmetic.

Corollary 2.7. There exists a model of **PA** that is not (isomorphic to) the standard model $\mathcal{N}$.

Proof (Proof)**.** Since $\mathbf{PA} \vdash A_{\mathbf{PA}}$, we know by the completeness theorem for predicate logic that $\mathbf{PA} \nvDash A_{\mathbf{PA}}$, i.e., there is a model $\mathcal{M}$ of **PA** such that $\mathcal{M} \nvDash A_{\mathbf{PA}}$. However, the standard model $\mathcal{N}$ is a model of $A_{\mathbf{PA}}$, in other words, $A_{\mathbf{PA}}$ is true in the intended interpretation. Therefore, $\mathcal{M}$ cannot be the standard model $\mathcal{N}$. ■

2.1 Proof of the First Incompleteness Theorem

Outline:

Step 1. **Gödel Numbering**: There is a coding function g that maps terms, formulae and proofs into natural numbers.

Step 2. **Fixed Point Lemma**: Given a formula A with a single free variable y, there exists a fixed point of the transformation which maps each sentence D to the sentence $A[g(D)/y]$.

Step 3. **Provability Predicate**: Consider the provability predicate $T_{\mathbf{T}}(y)$ which can be read as "there is a proof of the sentence $g^{-1}(y)$." The Fixed Point Lemma applied to $\neg \mathrm{Prov}_{\mathbf{T}}(y)$ yields a Gödel sentence saying that it is not provable. We then analyse which consistency assumptions suffice, and use Rosser's modification for the theorem as stated.

Step 1. Gödel Numbering

Definition 2.8 (Coding, Effectiveness)**.** A **coding** of **PA** is a total map g from the expressions of the syntax (terms, formulae, proofs) to $\mathbb{N}$ such that

1. g is injective;
2. g is recursive;
3. g^{-1} on the image of g is recursive, too.

Definition 2.9 (Gödel's coding function)**. Gödel's coding function** g on the expressions of the syntax (terms, formulae, proofs) is inductively defined as follows:[a]

Terms: Each code for a term is of the form

$$2 \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7};$$

the exponent e_3 tells whether the term is 0, a variable, a successor, a sum, or a multiplication:

0	t'	$t+s$	$t \cdot s$	a_i
1	2	3	4	$5+i$

the parameters of a term, i.e., the index of the variable, or the arguments of the successor, of the sum, or the multiplication, are the exponents e_5, e_7, in that order.

For instance:

$$g(t+s) = 2 \cdot 3^3 \cdot 5^{g(t)} \cdot 7^{g(s)}.$$

— **Formulae**: Each code for a formula is of the form

$$2^2 \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7};$$

the exponent e_3 tells which kind of formula the code represents:

$t = s$	$A \wedge B$	$A \vee B$	$A \to B$	$\forall x A$	$\exists x A$
1	2	3	4	5	6

the parameters of a formula are recorded by the exponents e_5, e_7 (the order is a matter of convention).

For instance:

$$g(A \wedge B) = 2^2 \cdot 3^2 \cdot 5^{g(A)} \cdot 7^{g(B)}, \qquad g(\exists x A) = 2^2 \cdot 3^6 \cdot 5^{g(A)} \cdot 7^{g(x)}.$$

— **Sequences**: The code for a finite sequence $\langle n_1, \ldots, n_k \rangle$ of natural numbers is:

$$2^3 \cdot \prod_{1 \leqslant i \leqslant k} p_{i+1}^{n_i+1},$$

with p_j the j-th prime number.

— **Proofs**: Each code for a proof is of the form

$$2^4 \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7} \cdot 11^{e_{11}} \cdot 13^{e_{13}} \cdot 17^{e_{17}} \cdot 19^{e_{19}} \cdot 23^{e_{23}};$$

the exponent e_3 tells which is the last rule applied:

$\wedge^+$	$\wedge_L^-$	$\wedge_R^-$	$\vee_L^+$	$\vee_R^+$	$\vee^-$	$\to^+$	$\to^-$	$\forall^+$	$\forall^-$	$\exists^+$	$\exists^-$	*raa*	Ax. i
1	2	3	4	5	6	7	8	9	10	11	12	13	$i+13$

the exponent e_5 tells the conclusion of such rule; the exponents e_7, e_{11}, e_{13} tell the derivations of the branchings; the exponent e_{17} tells the substitution term t in rules $\forall^-$ and $\exists^+$; the exponent e_{19} tells the sequence of codings of the bounded variables in the rules for quantifiers and in Axioms; the exponent e_{23} tells the sequence of codings of the open assumptions.

For instance:

$$g\left(\begin{array}{c}\pi_1\colon \vdots\ A \quad \pi_2\colon \vdots\ B\\ \hline A \wedge B\end{array} \wedge^+\right) = 2^4 \cdot 3^1 \cdot 5^{g(A\wedge B)} \cdot 7^{g(\pi_1)} \cdot 11^{g(\pi_2)}$$

$$g\left(\begin{array}{c}\pi_1\colon \begin{array}{c}C\\ \vdots\\ A\end{array} \quad \pi_2\colon \begin{array}{c}D\\ \vdots\\ B\end{array}\\ \hline A \wedge B\end{array} \wedge^+\right) = 2^4 \cdot 3^1 \cdot 5^{g(A\wedge B)} \cdot 7^{g(\pi_1)} \cdot 11^{g(\pi_2)} \cdot 23^{g(\langle C,D\rangle)}$$

$$g\left(\begin{array}{c}\pi\colon \vdots\\ \forall x A\\ \hline A[t/x]\end{array} \forall^-\right) = 2^4 \cdot 3^{10} \cdot 5^{g(A[t/x])} \cdot 7^{g(\pi)} \cdot 17^{g(t)} \cdot 19^{g(\langle g(x)\rangle)}$$

[a] Several other coding functions are possible; there is nothing special about ours.

Remark 2.10.

1. Thanks to the **Fundamental Theorem of Arithmetic**, which says that natural numbers admit a unique factorisation in primes, g is computable, injective, and g^{-1} is computable.
2. Hence, intuitively, one can write the following formulae in **PA**:[a]

$$\begin{aligned}\mathrm{isTerm}(n) &= \text{'}n \text{ is the code of a term'},\\ \mathrm{isForm}(n) &= \text{'}n \text{ is the code of a formula'},\\ \mathrm{isSeq}(n) &= \text{'}n \text{ is the code of a sequence'},\\ \mathrm{isProof}(n) &= \text{'}n \text{ is the code of a proof'}.\end{aligned}$$

   This can be formalised by showing that the set of codes for terms is recursive, so that the fact that all recursive functions, sets and relations are representable in **PA** yields the result.

[a] They could be written down in an explicit way, but they are cumbersome formulae.

Definition 2.11 (Numeral)**.** The **numeral** $\ulcorner E \urcorner$ of an expression (term, formula, sequence or proof) E is the formal numeral denoting the natural number $g(E)$ in the language of **PA**; the integer $g(E)$ and the term $\ulcorner E \urcorner$ are different objects.

Numerals allow to internalise the codes: we can, indirectly, speak of an expression by stating a property of its code.[1]

Step 2. Fixed Point Lemma

[1] As soon as the property does not rely on the value, but on the "meaning" of the code, this is a perfectly reasonable way to proceed.

Lemma 2.12 (Fixed Point Lemma)**.** Let $\mathbf{T}$ be a theory in which every (primitive) recursive function is representable, and let A be a formula such that $\mathrm{FV}(A) = \{y\}$. Then, there is a formula D_A such that $\mathrm{FV}(D_A) = \emptyset$ and

$$\mathbf{T} \vdash D_A \leftrightarrow A[\ulcorner D_A \urcorner / y].$$

Proof (Proof)**.** Fix an effective Gödel coding of expressions. Let $d(n)$ be the **diagonal substitution function**: if n codes a formula $B(x)$ with one free variable, $d(n)$ is the code of the sentence $B(\overline{n})$, obtained by substituting the numeral for n; assign an arbitrary fixed value on other inputs. This function is computable (indeed, with a standard coding, primitive recursive), so its graph is representable in arithmetic by a formula $D(x, z)$. In particular, whenever $d(n) = m$, arithmetic proves

$$\forall z \Big(D(\overline{n}, z) \leftrightarrow z = \overline{m} \Big).$$

For the given formula $A(y)$, introduce

$$B(x) := \exists z \Big(D(x, z) \wedge A(z) \Big).$$

Let $b = g(B)$ and let $G := B(\overline{b})$; then $g(G) = d(b)$. Representability of d gives

$$\begin{aligned} \mathbf{T} \vdash G &\leftrightarrow \exists z \Big(D(\overline{b}, z) \wedge A(z) \Big) \\ &\leftrightarrow A(\overline{d(b)}) \leftrightarrow A(\ulcorner G \urcorner). \end{aligned}$$

Taking $D_A := G$ proves the claim. ∎

Step 3. The provability predicate and Gödel's argument

Definition 2.13 (Formalised provability)**.** Fix an effective axiomatisation of $\mathbf{T}$. The relation $\mathrm{Prf}_{\mathbf{T}}(p, q)$ means that p codes a valid $\mathbf{T}$-proof whose last line is the sentence with code q. Proof checking is effective (using the chosen proof coding, including the witnesses needed to check axiom instances); hence this relation can be represented by an arithmetical formula, also denoted $\mathrm{Prf}_{\mathbf{T}}(p, q)$. Define

$$\mathrm{Prov}_{\mathbf{T}}(q) := \exists p \, \mathrm{Prf}_{\mathbf{T}}(p, q).$$

Thus $\mathrm{Prov}_{\mathbf{T}}(q)$ says that the sentence coded by q is provable in $\mathbf{T}$.

Remark 2.14. The displayed arithmetical predicate is not simply a test for a prime factor in the code of a proof: it must express the **exact** conclusion of a **valid** proof in the **specified theory**. The existence of an appropriate formula is what arithmetisation of syntax establishes.

By the Fixed Point Lemma there is a sentence $G_{\mathbf{T}}$ such that

$$\mathbf{T} \vdash G_{\mathbf{T}} \leftrightarrow \neg \mathrm{Prov}_{\mathbf{T}}(\ulcorner G_{\mathbf{T}} \urcorner). \tag{2.1}$$

Proof (What the Gödel sentence establishes)**.** If $\mathbf{T} \vdash G_{\mathbf{T}}$, some concrete integer n codes such a proof. Since this is a finite, checkable fact, arithmetic proves $\mathrm{Prf}_{\mathbf{T}}(\overline{n}, \ulcorner G_{\mathbf{T}} \urcorner)$, and hence $\mathbf{T} \vdash \mathrm{Prov}_{\mathbf{T}}(\ulcorner G_{\mathbf{T}} \urcorner)$. Equation (2.1) then makes $\mathbf{T}$ inconsistent. Therefore, if $\mathbf{T}$ is consistent, it cannot prove $G_{\mathbf{T}}$; the sentence is consequently true in the standard model.

If $\mathbf{T} \vdash \neg G_{\mathbf{T}}$, however, the equation gives only $\mathbf{T} \vdash \mathrm{Prov}_{\mathbf{T}}(\ulcorner G_{\mathbf{T}} \urcorner)$. One **cannot** infer $\mathbf{T} \vdash G_{\mathbf{T}}$ from this: a theory may prove an existential statement about proofs without any actual proof existing. Ruling this out for the present Σ_1 statement requires, for instance, 1-consistency (or Σ_1-soundness). Under that additional hypothesis $\neg G_{\mathbf{T}}$ is not provable either. ■

Step 4. Rosser's improvement

The consistency-only result in Theorem 2.2 follows from a slightly different fixed point. For a sentence R, write

$$\mathrm{Ros}_{\mathbf{T}}(\ulcorner R \urcorner) := \forall p \Big(\mathrm{Prf}_{\mathbf{T}}(p, \ulcorner R \urcorner) \to \exists q \leqslant p \; \mathrm{Prf}_{\mathbf{T}}(q, \ulcorner \neg R \urcorner) \Big).$$

The diagonal lemma gives a sentence $R_{\mathbf{T}}$ equivalent in $\mathbf{T}$ to this formula: it says that any proof of itself is accompanied by a proof of its negation with no larger code.

Proof (Conclusion of Theorem 2.2)**.** Suppose $\mathbf{T}$ is consistent. If it proves $R_{\mathbf{T}}$, choose a code n of such a proof. There is no proof of $\neg R_{\mathbf{T}}$, by consistency. The finite computation that verifies the first fact and checks the nonexistence of any such proof with code at most n can be carried out in **PA**. Thus $\mathbf{T}$ proves the negation of the right-hand side of Rosser's equivalence, and so proves $\neg R_{\mathbf{T}}$, a contradiction.

Conversely, suppose $\mathbf{T}$ proves $\neg R_{\mathbf{T}}$ and let m code that proof. Consistency excludes every proof of $R_{\mathbf{T}}$. For each of the finitely many $p < m$, **PA** can verify $\neg \mathrm{Prf}_{\mathbf{T}}(\overline{p}, \ulcorner R_{\mathbf{T}} \urcorner)$. For $p \geqslant m$, the actual proof with code m witnesses $\exists q \leqslant p \; \mathrm{Prf}_{\mathbf{T}}(q, \ulcorner \neg R_{\mathbf{T}} \urcorner)$. Combining the finite cases with the inequality $p \geqslant m$, arithmetic proves Rosser's formula, hence $\mathbf{T}$ proves $R_{\mathbf{T}}$, again a contradiction.

Therefore neither $R_{\mathbf{T}}$ nor its negation is provable. Since there is no actual proof of $R_{\mathbf{T}}$, its defining implication is true for every natural number, so $\mathscr{N} \vDash R_{\mathbf{T}}$. Take $A_{\mathbf{T}} := R_{\mathbf{T}}$. ■

Exercises

Exercise. Explain why the fact that a sentence is true in the standard model does not imply its provability in **PA**. Which theorem from first-order logic is used to construct a non-standard model of arithmetic?

Exercise. Describe how a finite proof can be represented by a natural number. Which parts of a formal proof must be checkable by an effective procedure?

Exercise. Let $G_{\mathbf{T}}$ satisfy (2.1). Show precisely why consistency implies $\mathbf{T} \nvdash G_{\mathbf{T}}$; identify the extra hypothesis used to conclude $\mathbf{T} \nvdash \neg G_{\mathbf{T}}$ from the original Gödel sentence.

Exercise. In Rosser's argument, identify the use of consistency separately in the cases where $R_{\mathbf{T}}$ is provable and where its negation is provable.

3

Relating incompleteness and consistency

3.1 Gödel's Second Incompleteness Theorem

Theorem 3.1 (Gödel's Second Incompleteness Theorem for Arithmetic)**.** Let $\mathbf{T}$ be a consistent and effectively axiomatised extension of **PA**. Write $Cons_{\mathbf{T}} := \neg \mathrm{Prov}_{\mathbf{T}}(\ulcorner 0 = 1 \urcorner)$ for the standard arithmetical consistency statement, based on a conventional proof predicate satisfying the Hilbert–Bernays–Löb derivability conditions. Then

$$\boxed{\mathbf{T} \nvdash Cons_{\mathbf{T}}.}$$

Idea of Proof. The proof builds on a **formalised** version of the fixed-point argument. Let $G_{\mathbf{T}}$ be the Gödel sentence from (2.1). For the standard provability predicate, the derivability conditions allow $\mathbf{T}$ to prove

$$Cons_{\mathbf{T}} \longrightarrow \neg \mathrm{Prov}_{\mathbf{T}}(\ulcorner G_{\mathbf{T}} \urcorner) \longrightarrow G_{\mathbf{T}}.$$

The first implication is the nontrivial formalised version of the consistency argument; it cannot be obtained merely by quoting the external First Incompleteness Theorem. If $\mathbf{T}$ proved $Cons_{\mathbf{T}}$, it would therefore prove $G_{\mathbf{T}}$, which contradicts its consistency, as established in the preceding chapter. Thus $\mathbf{T} \nvdash Cons_{\mathbf{T}}$. ■

Remark 3.2. Even before we'd ever heard of Gödel's Second Theorem, we wouldn't have been convinced of its consistency by a derivation of $Cons_{\mathbf{T}}$ inside $\mathbf{T}$. For we'd just note that if $\mathbf{T}$ were in fact inconsistent, we'd be able to derive any sentence we like in the theory—including a false statement of its own consistency.

The Second Theorem now shows that we would indeed be right not to trust a theory's announcement of its own consistency. For, if $\mathbf{T}$ entails $Cons_{\mathbf{T}}$, then the theory must in fact be **inconsistent**.

3.2 Consistency results

One may wonder if there are informative approaches to establish the consistency of **PA**.

Remark 3.3. A stronger background theory can prove the consistency of **PA**, but the resulting proof is a **relative** consistency argument: its reliability depends on the principles used in the background theory. The Second Incompleteness Theorem rules out an internal proof of the standard consistency sentence in **PA** itself. Gentzen instead used principles of transfinite induction whose proof-theoretic strength differs from ordinary induction in **PA**.

Remark 3.4. The precise choice of $Cons_{\mathrm{T}}$ matters. The Second Incompleteness Theorem concerns a standard arithmetisation of consistency, with a proof predicate satisfying the appropriate derivability conditions; it does not apply to every sentence that can be informally described as "asserting consistency". Artemov's discussion [11] is one starting point for this distinction. One should not infer, however, that a non-standard formula provable in **PA** would provide an ordinary finitistic consistency proof of **PA**.

Theorem 3.5 (Gentzen's Consistency Theorem)**.** $0 = 1$ cannot be proven from the axioms of **PA**.

Idea of Proof. Gentzen utilises a sequent calculus formulation of **PA**, where proofs are organized based on a "dependency" relation, implying that more complex proofs follow simpler ones. In this ordering, a proof by induction of $\forall x A(x)$ surpasses all the proofs of $A(0), A(1), A(2)$, and so on, relying on the correctness of these simpler results. Thus, according to Gentzen's approach, the proof by induction of $\forall x A(x)$ must appear infinitely later in the sequence, after all the proofs of the various $A(n)$.

Gentzen's crucial step involves conducting an induction along this transfinite ordering of proofs. The very basic proofs at the beginning of the sequence evidently do not lead to contradiction. It becomes clear that once the correctness of all preceding proofs in the sequence has been established, the specific proof in question is also correct. This is due to the chosen ordering, which ensures that the correctness of a proof depends on the correctness of certain earlier proofs. Consequently, through transfinite induction, we can infer the correctness of all proofs, including the desired consistency.

It is important to note that the **PA** proofs being ordered are finite objects themselves, and we can encode them using Gödel numbers in a familiar manner. Therefore, when arranging the proofs, we are essentially considering an ordering of numbers. Determining the precedence of one number over another in this ordering is not a mysterious process; a computation without open-ended searches can settle the matter. ■

Remark 3.6. What kind of transfinite ordering is involved here? Gentzen's ordering of possible proof trees in his sequent calculus for **PA** turns out to have the order type of

the ordinals less than ε_0, which is defined as the limit of the sequence

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \omega^{\omega^{\omega^{\omega}}}, \ldots$$

So, what Gentzen's proof needs is the assumption that a relatively modest amount of transfinite induction—induction up to ε_0—is legitimate.

So, we can indeed prove the consistency of **PA** by using a theory which is weaker in some respects and stronger in others.

3.3 The Kirby–Paris Theorem

Paris and Kirby gave a strictly mathematical example of an incompleteness in first-order Peano arithmetic, the **Kirby–Paris theorem**, which is mathematically simple and interesting and which does not require a numerical coding of logical notions. Another interesting example is the **Paris–Harrington theorem**.

Definition 3.7 (Pure base k representation)**.** The **pure base k representation of n** is the result of writing n as a sum of powers of k, then rewriting the various exponents of k themselves as sums of powers of k, then rewriting these new exponents as sums of powers of k, etc. (writing k^0 as simply 1).

Example.

— The pure base 2 representation of 266 is

$$266 = 2^{2^{2^1+1}} + 2^{2^1+1} + 2^1.$$

— The pure base 3 representation of 266 is

$$266 = 3^{3^1+1+1} + 3^{1+1} + 3^{1+1} + 3^1 + 1 + 1.$$

Definition 3.8 (Bump function)**.** To evaluate the **bump function** $B_k : \mathbb{N} \to \mathbb{N}$, take the pure base k representation of n; bump up every k to $k+1$; and then subtract 1 from the resulting number.

Example. Let's calculate, for example, $B_2(19)$. Start with the pure base 2 representation of 19:

$$2^{2^{2^1}} + 2^1 + 1.$$

Bump up the base:

$$3^{3^{3^1}} + 3^1 + 1.$$

Subtract 1:

$$3^{3^{3^1}} + 3^1 = 7,625,597,484,990.$$

Definition 3.9 (Goodstein sequence)**.** The **Goodstein sequence** for n is:

$$n, B_2(n), B_3(B_2(n)), B_4(B_3(B_2(n))), \ldots$$

We denote the k-th term of the sequence starting with n by n_k.

Example.

— The Goodstein sequences starting with 2:

$$\begin{aligned}
2_1 &= 2 = 2^1, \\
2_2 &= B_2(2_1) = 3^1 - 1 = 2 = 1 + 1, \\
2_3 &= B_3(2_2) = 1 + 1 - 1 = 1, \\
2_4 &= B_4(2_3) = 1 - 1 = 0.
\end{aligned}$$

— The Goodstein sequences starting with 3:

$$\begin{aligned}
3_1 &= 3 = 2^1 + 1, \\
3_2 &= B_2(3_1) = 3^1 + 1 - 1 = 3^1, \\
3_3 &= B_3(3_2) = 4^1 - 1 = 3 = 1 + 1 + 1, \\
3_4 &= B_4(3_3) = 1 + 1 + 1 - 1 = 2 = 1 + 1, \\
3_5 &= B_5(3_4) = 1 + 1 - 1 = 1, \\
3_6 &= B_6(3_5) = 1 - 1 = 0.
\end{aligned}$$

— The Goodstein sequences starting with 4:

$$\begin{aligned}
4_1 &= 4 = 2^{2^1}, \\
4_2 &= B_2(4_1) = 3^{3^1} - 1 = 3^{1+1} + 3^{1+1} + 3^1 + 3^1 + 1 + 1, \\
4_3 &= B_3(4_2) = 4^{1+1} + 4^{1+1} + 4^1 + 4^1 + 1, \\
4_4 &= B_4(4_3) = 5^{1+1} + 5^{1+1} + 5^1 + 5^1, \\
4_5 &= B_5(4_4) = 6^{1+1} + 6^{1+1} + 6^1 + 1 + 1 + 1 + 1 + 1, \\
4_6 &= B_6(4_5) = 7^{1+1} + 7^{1+1} + 7^1 + 1 + 1 + 1 + 1, \\
4_7 &= B_7(4_6) = 8^{1+1} + 8^{1+1} + 8^1 + 1 + 1 + 1, \\
4_8 &= B_8(4_7) = 9^{1+1} + 9^{1+1} + 9^1 + 1 + 1, \\
4_9 &= B_9(4_8) = 10^{1+1} + 10^{1+1} + 10^1 + 1, \\
4_{10} &= B_{10}(4_9) = 11^{1+1} + 11^{1+1} + 11^1, \\
4_{11} &= B_{11}(4_{10}) = 12^{1+1} + 12^{1+1} + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, \\
4_{12} &= B_{12}(4_{11}) = 13^{1+1} + 13^{1+1} + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, \\
&\vdots
\end{aligned}$$

The terms shown above give only the beginning of the Goodstein sequence starting at 4. Unlike the sequences starting at 2 and 3, it has an extraordinarily long continuation; in particular, it does not reach zero at the 94th term.

Theorem 3.10 (Goodstein)**.** For all n, the Goodstein sequence eventually terminates at zero.

Idea of Proof. Take the Goodstein sequence for n. Render its k-th term into its pure base $k+1$ representation as in our examples above (with each sum presented in descending order of exponents). Now consider the parallel sequence that you get by going through and replacing each base number by ω (the first infinite ordinal). For example, the parallel sequence to the Goodstein sequence for 3 starts

$$\omega^1 + 1, \omega^1, 1 + 1 + 1, 1 + 1, 1, 0.$$

It is not hard to show that this parallel sequence of ordinals will in every case be strictly decreasing. But there just cannot be an infinite descending chain of such ordinals—that is a quite fundamental theorem about ordinals. Hence the ordinal sequence must terminate. And therefore the parallel Goodstein sequence for n must terminate too. ■

Theorem 3.11 (Kirby–Paris)**.** Goodstein's Theorem is not provable in **PA**.[a]

[a] Assuming that **PA** is consistent.

Idea of Proof. Goodstein's Theorem depends on the fundamental fact that there cannot be an infinite decreasing chain of ordinals which are sums of powers of ω, i.e. there cannot be an infinite decreasing chain of ordinals less than ε_0, the first ordinal that comes after all the sums of powers of ω. Proving that fundamental result is equivalent to showing that transfinite induction up to ε_0 is sound.

However, there are natural Gödel-numberings for the ordinals which are sums of powers of ω; so we can transmute claims about these ordinals into arithmetical claims about their numerical codes. So being able to prove Goodstein's theorem inside **PA** would be tantamount to **PA**'s being able to handle (via our codings) transfinite induction up to ε_0.

The non-provability result requires a substantive proof-theoretic analysis: Kirby and Paris show that a suitable uniform formalisation of Goodstein termination yields, over weak arithmetic, enough transfinite induction to entail the standard consistency of **PA**. By Gentzen's result (3.5) and the Second Incompleteness Theorem (3.1), this uniform statement cannot be proved in **PA** if that theory is consistent. This paragraph is an outline of the connection, not a full proof of the Kirby–Paris reduction. ■

Exercises

Exercise. Compute

1. $B_7(22)$,
2. $B_7(20)$,
3. $B_5(125)$,
4. $B_2(1024)$,
5. $B_{10}(1024)$.

Bibliography

Main references

[1] Abrusci, V.M. & Tortora de Falco, L. *Logica: Volume 2 – Incompletezza, teoria assiomatica degli insiemi*. (Springer, 2018)

[2] Barwise, J. (ed.). *Handbook of Mathematical Logic*. (North-Holland, 1977).

[3] Bell, J. & Machover, M. *A Course in Mathematical Logic*. (North-Holland, 1977)

[4] Kennedy, J. *Gödel's Incompleteness Theorems*. (Cambridge, 2022)

[5] Lolli, G. *La guerra dei trent'anni (1900-1930). Da Hilbert a Gödel*. (ETS, 2011)

[6] Masini, A. *Logica simbolica*. (McGraw-Hill, 2023)

[7] von Plato, J. *Can Mathematics Be Proved Consistent?: Gödel's Shorthand Notes & Lectures on Incompleteness* . (Springer, 2020)

[8] von Plato, J. *The Great Formal Machinery Works: Theories of Deduction and Computation at the Origins of the Digital Age*. (Princeton, 2017)

[9] Smith, P. *An Introduction to Gödel's Theorems. Second Edition*. (Cambridge, 2013)

[10] de Swart, H. *Philosophical and Mathematical Logic*. (Springer, 2014)

Historical sources

Here we only list the original works which are the heart of this course. For a more detailed bibliography, see, e.g. [9] and [8].

[11] Artemov, S. The Provability of Consistency. (2020)

[12] Gentzen, G. Die Widerspruchsfreiheit der reinen Zahlentheorie, *Mathematische Annalen*. **112**, 493–565 (1936).

[13] Gentzen, G. Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie, *Mathematische Annalen*. **119**, 149–161 (1943).

[14] Gödel, K. 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I'. *Monatshefte Für Mathematik Und Physik*. **38**, 173–198 (1931).

[15] Hilbert, D. & Ackermann, W. *Grundzüge der theoretischen Logik*. (Springer, 1928) Second edition 1938, third edition 1949, fourth edition 1959.

[16] Hilbert, D. & Bernays, P. *Grundlagen der Mathematik I*. (Springer, 1934)

[17] Hilbert, D. & Bernays, P. *Grundlagen der Mathematik II*. (Springer, 1939)

[18] Kirby, L. & Paris, J. Accessible Independence Results for Peano Arithmetic. *Bulletin of the London Mathematical Society*. **14**(4), 285–293 (1982).

[19] Paris, J. & Harrington, L. A mathematical incompleteness in Peano Arithmetic. In [2]: 1133–1142 (1977).

[20] Peano, G. *Arithmetices Principia Novo Methodo Exposita*. (Fratres Bocca, 1889)
[21] Rosser, J.B. 'Extensions of some theorems of Gödel and Church', *Journal of Symbolic Logic*. **1**, 87–91 (1936).

Further readings

[22] Bruni, R. *Kurt Gödel, un profilo*. (Carrocci, 2015)
[23] Hofstadter, D. *Gödel, Escher, Bach: an Eternal Golden Braid*. (Basic Books, 1979)
[24] Leonesi, S. & Toffalori, C. *Matematica, miracoli e paradossi. Storie di cardinali da Cantor a Gödel*. (Mondadori, 2007)
[25] Lolli, G. *Sotto il segno di Gödel*. (Il Mulino, 2007)
[26] Lolli, G. *Da Euclide a Gödel*. (Il Mulino, 2010)
[27] Lolli, G. *I teoremi di incompletezza*. (Il Mulino, 2019)
[28] Lolli, G. & Pagallo, U. (eds) *La complessità di Gödel*. (Giappichelli, 2008)
[29] Odifreddi, P. *Il diavolo in cattedra: La logica da Aristotele a Gödel*. (Einaudi, 2017)
[30] Odifreddi, P. *Il dio della logica. Vita geniale di Kurt Gödel, matematico della filosofia*. (TEA, 2019)
[31] von Plato, J. *Portrait of Young Gödel: Education, First Steps in Logic, the Problem of Completeness*. (Springer, 2024)